Class Exercise

1. **Prove the following limits:** (a) $\lim_{z\to z_0} \mathrm{Re}\ z = \mathrm{Re}\ z_0$;
   (b) $\lim_{z\to z_0} \overline{z} = \overline{z_0}$;
   (c) If $\lim_{z\to z_0} f(z) = w_0$, then $\lim_{z\to z_0} |f(z)| = |w_0|$.

2. **Show that the derivative $f'(z)$ does not exist at any point z for the following functions:**
   (a) $f(z) = \mathrm{Re}(z)$;
   (b) $f(z) = \mathrm{Im}(z)$.

3. **Evaluate the following limits:**
   (c) $\lim_{z\to z_0} \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials, and $Q(z_0) \neq 0$.

1. **Prove the following limits:** (a) $\lim_{z \to z_0} \text{Re } z = \text{Re } z_0$;
   (b) $\lim_{z \to z_0} \overline{z} = \overline{z_0}$;
   (c) If $\lim_{z \to z_0} f(z) = w_0$, then $\lim_{z \to z_0} |f(z)| = |w_0|$.

Sol:

(a)

The $\delta - \epsilon$ definition states:

$$\lim_{z \to z_0} f(z) = f(z_0) \quad \text{if for every } \epsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Here, $f(z) = \text{Re}\, z$, and $|f(z) - f(z_0)| = |\text{Re}\, z - \text{Re}\, z_0|$.

Since $|\text{Re}\, z - \text{Re}\, z_0| \leq |z - z_0|$ (a basic property of the real part of a complex number), choosing $\delta = \epsilon$ guarantees that:

$$|z - z_0| < \delta \implies |\text{Re}\, z - \text{Re}\, z_0| < \epsilon.$$

Thus, by the $\delta - \epsilon$ definition,

$$\lim_{z \to z_0} \text{Re}\, z = \text{Re}\, z_0.$$

(b)

The complex conjugate of z is given by $\overline{z} = x - iy$, where $z = x + iy$. We aim to prove that:

$$\lim_{z \to z_0} \overline{z} = \overline{z_0}.$$

From the $\delta - \epsilon$ definition, we need to show:

$$|z - z_0| < \delta \implies |\overline{z} - \overline{z_0}| < \epsilon.$$

The distance between $\overline{z}$ and $\overline{z_0}$ is:

$$|\overline{z} - \overline{z_0}| = |(x - iy) - (x_0 - iy_0)| = |(x - x_0) - i(y - y_0)|.$$

By the definition of the modulus of a complex number:

$$|\overline{z} - \overline{z_0}| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - z_0|.$$

Thus, choosing $\delta = \epsilon$ guarantees that:

$$|z - z_0| < \delta \implies |\overline{z} - \overline{z_0}| < \epsilon.$$

Therefore, by the $\delta - \epsilon$ definition:

$$\lim_{z \to z_0} \overline{z} = \overline{z_0}.$$

(c)

We are given that $\lim_{z \to z_0} f(z) = w_0$, which means:

$$\text{For every } \epsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon.$$

The key inequality is:

$$\big||f(z)| - |w_0|\big| \leq |f(z) - w_0|.$$

This tells us that the distance between $|f(z)|$ and $|w_0|$ is at most the distance between $f(z)$ and w_0. Therefore:

$$|z - z_0| < \delta \implies \big||f(z)| - |w_0|\big| \leq |f(z) - w_0| < \epsilon.$$

This means that for every $\epsilon > 0$, there exists a $\delta > 0$ (the same δ as for $|f(z) - w_0| < \epsilon$) such that:

$$|z - z_0| < \delta \implies \big||f(z)| - |w_0|\big| < \epsilon.$$

2. **Show that the derivative $f'(z)$ does not exist at any point z for the following functions:**
   (a) $f(z) = \mathrm{Re}(z)$;
   (b) $f(z) = \mathrm{Im}(z)$.

Sol:

(a)

The derivative is defined as:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z},$$

where:

$$\Delta w = f(z + \Delta z) - f(z).$$

For $f(z) = \mathrm{Re}(z)$, the real part of $z = x + iy$ is $\mathrm{Re}(z) = x$. Therefore:

$$\Delta w = \mathrm{Re}(z + \Delta z) - \mathrm{Re}(z).$$

Let $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$. Then:

$$\mathrm{Re}(z + \Delta z) = \mathrm{Re}((x + \Delta x) + i(y + \Delta y)) = x + \Delta x.$$

Thus:

$$\Delta w = \mathrm{Re}(z + \Delta z) - \mathrm{Re}(z) = (x + \Delta x) - x = \Delta x.$$

The derivative becomes:

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta z}.$$

Now, $\Delta z = \Delta x + i\Delta y$. We evaluate $\frac{\Delta x}{\Delta z}$ along different paths.

1. **Horizontal path ($\Delta z = \Delta x + i0$):**
   - Here, $\Delta z = \Delta x$, so:

   $$\frac{\Delta x}{\Delta z} = \frac{\Delta x}{\Delta x} = 1.$$

2. **Vertical path ($\Delta z = 0 + i\Delta y$):**
   - Here, $\Delta z = i\Delta y$, so:

   $$\frac{\Delta x}{\Delta z} = \frac{\Delta x}{i\Delta y}.$$

   Since $\Delta x = 0$ along the vertical path, we have:

   $$\frac{\Delta x}{\Delta z} = 0.$$

Since the limit depends on the path, the derivative $f'(z)$ does **not exist** at any point z for $f(z) = \mathrm{Re}(z)$.

(b)

For $f(z) = \mathrm{Im}(z)$, the imaginary part of $z = x + iy$ is $\mathrm{Im}(z) = y$. Then:

$$\Delta w = \mathrm{Im}(z + \Delta z) - \mathrm{Im}(z).$$

Let $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$. Then:

$$\mathrm{Im}(z + \Delta z) = \mathrm{Im}((x + \Delta x) + i(y + \Delta y)) = y + \Delta y.$$

Thus:

$$\Delta w = \mathrm{Im}(z + \Delta z) - \mathrm{Im}(z) = (y + \Delta y) - y = \Delta y.$$

The derivative becomes:

$$\frac{\Delta w}{\Delta z} = \frac{\Delta y}{\Delta z}.$$

Now, $\Delta z = \Delta x + i\Delta y$. We evaluate $\frac{\Delta y}{\Delta z}$ along different paths.

1. **Horizontal path ($\Delta z = \Delta x + i0$):**
   - Here, $\Delta z = \Delta x$, so:

     $$\frac{\Delta y}{\Delta z} = \frac{\Delta y}{\Delta x}.$$

     Since $\Delta y = 0$ along the horizontal path, we have:

     $$\frac{\Delta y}{\Delta z} = 0.$$

2. **Vertical path ($\Delta z = 0 + i\Delta y$):**
   - Here, $\Delta z = i\Delta y$, so:

     $$\frac{\Delta y}{\Delta z} = \frac{\Delta y}{i\Delta y}.$$

     Simplify:

     $$\frac{\Delta y}{\Delta z} = \frac{\Delta y}{i\Delta y} = \frac{1}{i} = -i.$$

Since the limit depends on the path, the derivative $f'(z)$ does **not exist** at any point z for $f(z) = \mathrm{Im}(z)$.

3. **Evaluate the following limits:**

(c) $\lim_{z \to z_0} \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials, and $Q(z_0) \neq 0$.

Sol:

(c)

1. **Given:**
   - $P(z)$ and $Q(z)$ are polynomials, and $Q(z_0) \neq 0$.
   - The function is $f(z) = \frac{P(z)}{Q(z)}$, and we need to evaluate its limit as $z \to z_0$.
2. **Using Theorem 2:**
   - The limit of a quotient is the quotient of the limits (if the denominator is not zero).
   - Polynomials are continuous, so $\lim_{z \to z_0} P(z) = P(z_0)$ and $\lim_{z \to z_0} Q(z) = Q(z_0)$.
3. **Step-by-step:**

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}.$$
